

The competition numbers of ternary Hamming graphs

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Abstract

It is known to be a hard problem to compute the competition number $k(G)$ of a graph G in general. Park and Sano [13] gave the exact values of the competition numbers of Hamming graphs $H(n, q)$ if $1 \leq n \leq 3$ or $1 \leq q \leq 2$. In this paper, we give an explicit formula of the competition numbers of ternary Hamming graphs.

Keywords: competition graph; competition number; edge clique cover; Hamming graph

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1. Introduction

The notion of a competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space. The *competition graph* $C(D)$ of a digraph D is a (simple undirected) graph which has the same vertex set as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D . For any graph G , G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Roberts [15] defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph. Opsut [9] showed that the computation of the competition number of a graph is an NP-hard problem (see [3] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to compute the competition numbers for various graph classes (see [4], [5], [6], [7], [8], [10], [11], [12], [14], [16], [17], for recent research). For some special graph families, we have explicit formulas for computing competition numbers. For example, if G is a chordal graph without isolated vertices then $k(G) = 1$, and if G is a nontrivial triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [15]).

In this paper, we study the competition numbers of Hamming graphs. For a positive integer q , we denote a q -set $\{1, 2, \dots, q\}$ by $[q]$. Also we denote the set of n -tuple over $[q]$ by $[q]^n$. For positive integers n and q , the *Hamming graph* $H(n, q)$ is the graph which has the vertex set $[q]^n$ and in which two vertices x and y are adjacent if $d_H(x, y) = 1$, where $d_H : [q]^n \times [q]^n \rightarrow \mathbb{Z}$ is the *Hamming distance* defined by $d_H(x, y) := |\{i \in [n] \mid x_i \neq y_i\}|$. Note that the diameter of the Hamming graph $H(n, q)$ is equal to n if $q \geq 2$ and that the number of edges of the Hamming graph $H(n, q)$ is equal to $\frac{1}{2}n(q-1)q^n$. Park and Sano [13] gave the exact values of the competition numbers of Hamming graphs $H(n, q)$ if $1 \leq n \leq 3$ or $1 \leq q \leq 2$ as follows. For $n \geq 1$, $H(n, 1)$ has no edge and so $k(H(n, 1)) = 0$. For $n \geq 1$, $H(n, 2)$ is a n -cube and $k(H(n, 2)) = (n-2)2^{n-1} + 2$. For $q \geq 2$, $H(1, q)$ is complete graph and so $k(H(1, q)) = 1$.

Theorem 1.1 ([13]). *For $q \geq 2$, we have $k(H(2, q)) = 2$.*

Theorem 1.2 ([13]). *For $q \geq 3$, we have $k(H(3, q)) = 6$.*

In this paper, we give the exact values of the competition numbers of ternary Hamming graphs $H(n, 3)$. Our main result is the following:

Theorem 1.3. *For $n \geq 3$, we have*

$$k(H(n, 3)) = (n-3)3^{n-1} + 6.$$

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2. Preliminaries

We use the following notation and terminology in this paper. For a digraph D , a sequence v_1, v_2, \dots, v_n of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D . For a digraph D and a vertex v of D , we define the *in-neighborhood* $N_D^-(v)$ of v in D to be the set $\{w \in V(D) \mid (w, v) \in A(D)\}$, and a vertex in $N_D^-(v)$ is called an *in-neighbor* of v in D . For a graph G and a nonnegative integer k , we denote by $G \cup I_k$ the graph such that $V(G \cup I_k) = V(G) \cup I_k$ and $E(G \cup I_k) = E(G)$, where I_k is a set of k isolated vertices.

For a clique S of a graph G and an edge e of G , we say e is *covered* by S if both of the endpoints of e are contained in S . An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G . An edge clique cover of G is called a *minimum edge clique cover* of G if its size is equal to $\theta_E(G)$.

Let $\pi_j : [q]^n \rightarrow [q]^{n-1}$ be a map defined by $(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. For $j \in [n]$ and $p \in [q]^{n-1}$, we let

$$S_j(p) := \pi_j^{-1}(p) = \{x \in [q]^n \mid \pi_j(x) = p\}. \quad (1)$$

Note that $S_j(p)$ is a clique of $H(n, q)$ with size q . Let

$$\mathcal{F}(n, q) := \{S_j(p) \mid j \in [n], p \in [q]^{n-1}\}. \quad (2)$$

Then $\mathcal{F}(n, q)$ is the family of maximal cliques of $H(n, q)$. Park and Sano [13] showed the following:

Lemma 2.1 ([13]). *The following hold:*

- Let $n \geq 2$ and $q \geq 2$, and let K be a clique of $H(n, q)$ with size at least 2. Then there exists a unique maximal clique S of $H(n, q)$ containing K .
- The edge clique cover number of $H(n, q)$ is equal to nq^{n-1} .
- Any minimum edge clique cover of $H(n, q)$ consists of edge disjoint maximum cliques.
- The family $\mathcal{F}(n, q)$ defined by (2) is a minimum edge clique cover of $H(n, q)$.

Now we present the following lemma:

Lemma 2.2. *Let G be a graph. Suppose that any edge of G is contained in exactly one maximal clique of G . Let \mathcal{F} be the family of all maximal cliques of size at least two in G , and let k be an integer with $k \geq k(G)$. Then there exists an acyclic digraph D satisfying the following:*

- The competition graph of D is $G \cup I_k$.
- For any vertex v in D , $N_D^-(v) \in \mathcal{F} \cup \{\emptyset\}$.
- The number of vertices which have no in-neighbor in D is equal to $k + |V(G)| - |\mathcal{F}|$.

Proof. Let k be an integer such that $k \geq k(G)$. By the definition of the competition number of a graph, there exists an acyclic digraph satisfying (a). Suppose that any acyclic digraphs satisfying (a) does not satisfies (b). Let D be an acyclic digraph which maximizes $|\{v \in V(D) \mid N_D^-(v) = \emptyset\}|$ among all acyclic digraphs satisfying (a) but not (b). Since D does not satisfy (b), there exists a vertex v^* in D such that $N_D^-(v^*) \notin \mathcal{F} \cup \{\emptyset\}$. Since D maximizes $|\{v \in V(D) \mid N_D^-(v) = \emptyset\}|$, we may assume that $|N_D^-(v)| \neq 1$ for any $v \in V(D)$. By the assumption that any edge of G is contained in exactly one maximal clique of G , any clique of size at least two is contained in a unique maximal clique in G . Since $N_D^-(v^*)$ is a clique of size at least two, there exists a unique maximal clique $S \in \mathcal{F}$ containing $N_D^-(v^*)$.

Let $\sigma := (v_1, v_2, \dots, v_{|V(D)|})$ be an acyclic ordering of D , i.e., if $(v_i, v_j) \in A(D)$ then $i < j$. Let v_i and v_j be two vertices of S whose indices are largest in the acyclic ordering σ . Since v_i and v_j are adjacent in G , there is a vertex $v_l \in V(D)$ such that (v_i, v_l) and (v_j, v_l) are arcs of D . Then we define a digraph D_1 by $V(D_1) = V(D)$ and

$$A(D_1) = (A(D) \setminus \{(x, v^*) \mid x \in N_D^-(v^*)\}) \cup \{(x, v_l) \mid x \in S\}.$$

Then the digraph D_1 is acyclic, since the index l of v_l is larger than the index of any vertex in S . By the assumption that any edge of G is contained in exactly one maximal clique of G , we have $N_D^-(v_l) \subseteq S$, which implies that $C(D_1) = G \cup I_k$. Since $|\{v \mid N_D^-(v) = \emptyset\}| > |\{v \mid N_{D_1}^-(v) = \emptyset\}|$, we reach a contradiction to the choice of D . Thus, there exists an acyclic digraph satisfying both (a) and (b).

Let D_2 be an acyclic digraph satisfying both (a) and (b). If the digraph D_2 has two vertices u, w such that $N_{D_2}^-(u) = N_{D_2}^-(w) \neq \emptyset$, then we can obtain an acyclic digraph D_3 such that all the nonempty in-neighborhoods of vertices in D_3 are distinct by deleting arcs in $\{(x, u) \mid x \in N_{D_2}^-(u)\}$ from D_2 . Therefore, we may assume that all the nonempty in-neighborhoods $N_{D_3}^-(v)$ are distinct. Then the number of vertices v such that $N_{D_3}^-(v) \neq \emptyset$ is exactly equal to $|\mathcal{F}|$. Therefore, $|\{v \in V(D_3) \mid N_{D_3}^-(v) = \emptyset\}| = |V(D_3)| - |\mathcal{F}| = k + |V(G)| - |\mathcal{F}|$. Thus the digraph D_3 satisfies (a), (b), and (c). Hence the lemma holds. \square

By Lemmas 2.1 and 2.2, the following holds.

Corollary 2.3. *Let k be an integer with $k \geq k(H(n, q))$. Then there exists an acyclic digraph D satisfying the following:*

- (a) $C(D) = H(n, q) \cup I_k$.
- (b) $N_D^-(v) \in \mathcal{F}(n, q) \cup \{\emptyset\}$ for any $v \in V(D)$.
- (c) $|\{v \in V(D) \mid N_D^-(v) = \emptyset\}| = k - (n - q)q^{n-1}$.

3. Proof of Theorem 1.3

In this section, we present the proof of the main result. The following theorem gives an upper bound for the competition numbers of Hamming graphs $H(n, q)$ where $2 \leq q \leq n$.

Theorem 3.1. *For $2 \leq q \leq n$, we have*

$$k(H(n, q)) \leq (n - q)q^{n-1} + k(H(q, q)).$$

Proof. We prove the theorem by induction on n . If $n = 2$, then $n = q = 2$ and the theorem trivially holds. For simplicity, for any $2 \leq q \leq n$, let $\alpha(n, q) := (n - q)q^{n-1} + k(H(q, q))$. Let $n \geq 3$ and we assume that $k(H(n - 1, q)) \leq \alpha(n - 1, q)$ holds for any q such that $2 \leq q \leq n - 1$. Now consider a Hamming graph $H(n, q)$ where q is an integer satisfying $2 \leq q \leq n$. If $n = q$, then the theorem clearly holds. Suppose that $2 \leq q \leq n - 1$. For $i \in [q]$, let $H^{(i)}$ be the subgraph of $H(n, q)$ induced by a vertex set $\{(x_1, x_2, \dots, x_n) \in [q]^n \mid x_n = i\}$. Then each $H^{(i)}$ is isomorphic to the Hamming graph $H(n - 1, q)$.

We denote the minimum edge clique cover of $H^{(i)}$ by $\mathcal{F}^{(i)}(n - 1, q)$. By the induction hypothesis, it holds that $k(H^{(i)}) = k(H(n - 1, q)) \leq \alpha(n - 1, q)$. By Corollary 2.3, there exists an acyclic digraph $D^{(i)}$ for each $i \in [q]$ such that

- (a) $C(D^{(i)}) = H^{(i)} \cup I_{\alpha(n-1, q)}^{(i)}$,
- (b) $N_{D^{(i)}}^-(v) \in \mathcal{F}^{(i)}(n - 1, q) \cup \{\emptyset\}$ for any $v \in V(D^{(i)})$,
- (c) $|\{v \in V(D^{(i)}) \mid N_{D^{(i)}}^-(v) = \emptyset\}| = \alpha(n - 1, q) - (n - 1 - q)q^{n-2} = k(H(q, q))$.

where $I_{\alpha(n-1, q)}^{(i)}$ is a set of $\alpha(n - 1, q)$ isolated vertices. Then by (c), we may let, for each $i \in [q]$,

$$W^{(i)} := \{v \in V(D^{(i)}) \mid N_{D^{(i)}}^-(v) = \emptyset\} = \{w_1^{(i)}, \dots, w_{k(H(q, q))}^{(i)}\}.$$

Since $q \leq n - 1$, it holds that $k(H(q, q)) = \alpha(n - 1, q) - ((n - 1) - q)q^{n-2} \leq \alpha(n - 1, q)$. By (a), there are at least $k(H(q, q))$ isolated vertices in $I_{\alpha(n-1, q)}^{(i)}$ of $C(D^{(i)})$. Let $J^{(i)} := \{j_1^{(i)}, \dots, j_{k(H(q, q))}^{(i)}\}$ be a set of $k(H(q, q))$ vertices which belong to $I_{\alpha(n-1, q)}^{(i)}$ in $C(D^{(i)})$. Let D be the digraph defined by

$$\begin{aligned} V(D) &:= V(D^{(1)}) \cup \bigcup_{i=2}^q (V(D^{(i)}) \setminus J^{(i)}), \\ A(D) &:= A(D^{(1)}) \cup \bigcup_{i=2}^q (A(D^{(i)}) \setminus \{(x, j_l^{(i)}) \mid 1 \leq l \leq k(H(q, q)), x \in N_{D^{(i)}}^-(j_l^{(i)})\}) \\ &\quad \cup \bigcup_{i=2}^q \{(x, w_l^{(i-1)}) \mid 1 \leq l \leq k(H(q, q)), x \in N_{D^{(i)}}^-(j_l^{(i)})\}. \end{aligned}$$

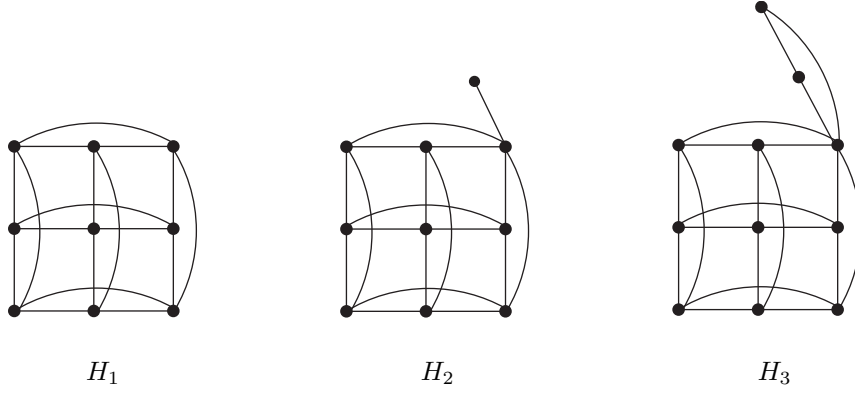


Figure 1: Induced subgraphs of $H(n, 3)$

Since each digraph $D^{(i)}$ is acyclic, the digraph D is also acyclic. In addition, $C(D) = (\bigcup_{i=1}^q H^{(i)}) \cup I_k$ where $k := q \cdot \alpha(n-1, q) - (q-1) \cdot k(H(q, q))$.

Note that by the definition of $H^{(i)}$, the graph $\bigcup_{i=1}^q H^{(i)}$ is a spanning subgraph of $H(n, q)$ except the edges of cliques in $\{S_n(x) \mid x \in [q]^{n-1}\}$. Therefore, by letting D^* be the digraph defined by

$$\begin{aligned} V(D^*) &:= V(D) \cup I_{q^{n-1}} = V(D) \cup \{z_x \mid x \in [q]^{n-1}\}, \\ A(D^*) &:= A(D) \cup \bigcup_{x \in [q]^{n-1}} \{(y, z_x) \mid y \in S_n(x)\}, \end{aligned}$$

we obtain an acyclic digraph D^* such that $C(D^*) = H(n, q) \cup I_{k^*}$, where

$$\begin{aligned} k^* &:= k + q^{n-1} \\ &= q \cdot \alpha(n-1, q) - (q-1) \cdot k(H(q, q)) + q^{n-1} \\ &= q \cdot ((n-1-q)q^{n-2} + k(H(q, q))) - (q-1) \cdot k(H(q, q)) + q^{n-1} \\ &= (n-q)q^{n-1} + k(H(q, q)) = \alpha(n, q). \end{aligned}$$

Therefore, $k(H(n, q)) \leq \alpha(n, q)$. Hence, the theorem holds. \square

The following corollary follows from Theorem 3.1.

Corollary 3.2. For $n \geq 3$, we have $k(H(n, 3)) \leq (n-3)3^{n-1} + 6$.

In the following, we show a lower bound for the competition numbers of ternary Hamming graphs $H(n, 3)$ where $n \geq 3$.

Lemma 3.3. Let $n \geq 3$, and let G be a subgraph of the ternary Hamming graph $H(n, 3)$ with 10 vertices. Then the number of triangles in G is at most 6. Moreover, it is exactly equal to 6 if and only if G is isomorphic to either $H_1 \cup I_1$ or H_2 in Figure 1.

Proof. We denote by t_G the number of triangles in a graph G . For a vertex v in a graph G , we denote by $t_G(v)$ the number of triangles in G containing the vertex v . For any graph G , it holds that

$$\sum_{v \in V(G)} t_G(v) = 3 \times t_G. \quad (3)$$

Let G be a subgraph of $H(n, 3)$ with 10 vertices such that t_G is the maximum among all the subgraphs of $H(n, 3)$ with 10 vertices. Since the graph H_2 drawn in Figure 1 is a subgraph of $H(n, 3)$ with 10 vertices and 6 triangles, we have $t_G \geq 6$. Let $v_1, v_2, v_3, \dots, v_9, v_{10}$ be the vertices of G .

Suppose that there exists a vertex v_i such that $t_G(v_i) \geq 3$. Then we will reach a contradiction. Without loss of generality, we may assume that $t_G(v_1) \geq 3$ and that $\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_1, v_6, v_7\}$ are cliques which contain v_1 . Since each edge is contained exactly one triangle by Lemma 2.1, then the subgraph of $H(n, 3)$ induced by $\{v_1, v_2, \dots, v_7\}$ is isomorphic to the graph H_4 drawn in Figure 2.

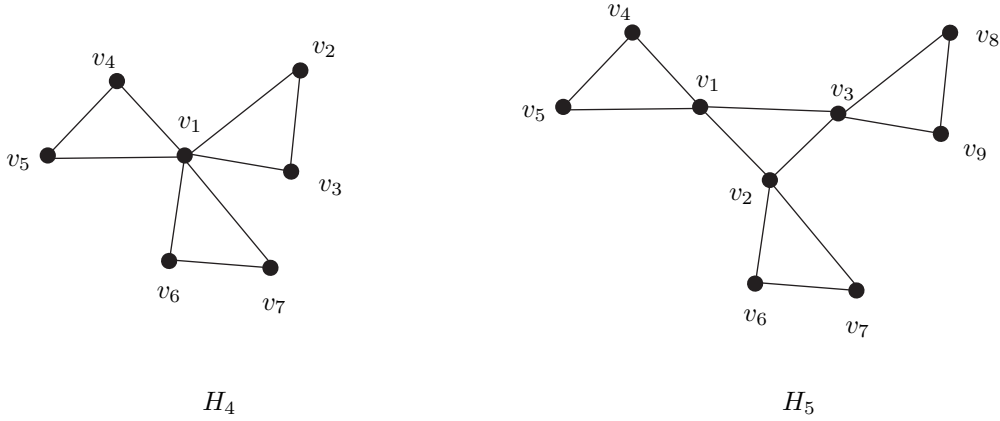


Figure 2: Subgraphs of $H(n, 3)$

Since G has at least 6 triangles, there exist at least three triangles T_1, T_2, T_3 in G which are not in the induced subgraph H_4 . For each $1 \leq i \leq 3$, the triangle T_i has at least two vertices in $\{v_8, v_9, v_{10}\}$ since each edge of G is contained in exactly one triangle. We may assume that $T_1 \supseteq \{v_8, v_9\}$, $T_2 \supseteq \{v_8, v_{10}\}$, and $T_3 \supseteq \{v_9, v_{10}\}$. Then $\{v_8, v_9, v_{10}\}$ forms a triangle, which contradicts the fact that each edge of G is contained exactly one triangle. Therefore, $t_G(v_i) \leq 2$ for all $v_i \in V(G)$. Since $t_G(v_i) \leq 2$ for all $v_i \in V(G)$, it holds that $\sum_{i=1}^{10} t_G(v_i) \leq 20$. By (3), $\sum_{i=1}^{10} t_G(v_i)$ must be a multiple of 3, $\sum_{i=1}^{10} t_G(v_i) \leq 18$. Since $t_G \geq 6$, we also have $\sum_{i=1}^{10} t_G(v_i) \geq 18$ by (3). Thus, we have $\sum_{i=1}^{10} t_G(v_i) = 18$ and $t_G = 6$.

Since $\sum_{i=1}^{10} t_G(v_i) = 18$ and $t_G(v_i) \leq 2$ for all $v_i \in V(G)$, without loss of generality, we may assume that $t_G(v_{10}) \leq 1$. Let $G - v_{10}$ be the graph obtained from G by deleting v_{10} . To show the “moreover” part, it is sufficient to show that $G - v_{10}$ is isomorphic to the graph $H_1 (\cong H(2, 3))$ drawn in Figure 1.

Note that $15 \leq \sum_{i=1}^9 t_{G-v_{10}}(v_i) \leq 18$, and $t_{G-v_{10}}(v) \leq 2$ for all $v \in V(G - v_{10})$. Then $G - v_{10}$ has at least 6 vertices v such that $t_{G-v_{10}}(v) = 2$, and so $G - v_{10}$ has a triangle $T_0 = \{v_1, v_2, v_3\}$ such that $t_{G-v_{10}}(v_i) = 2$ for any $i = 1, 2, 3$. For $i = 1, 2, 3$, let T_i be the triangle containing v_i and $T_i \neq T_0$. Then the triangles T_1, T_2 , and T_3 are mutually vertex disjoint, and we may assume that $T_1 = \{v_1, v_4, v_5\}$, $T_2 = \{v_2, v_6, v_7\}$, and $T_3 = \{v_3, v_8, v_9\}$ (see the graph H_5 drawn in Figure 2). Since $G - v_{10}$ has at least 6 vertices v such that $t_{G-v_{10}}(v) = 2$, we may assume that $t_{G-v_{10}}(v_4) = 2$. Then the triangle containing v_4 in $G - v_{10}$, which is not T_1 , are containing one of $\{v_6, v_7\}$ and one of $\{v_8, v_9\}$. Without loss of generality, we may assume that $\{v_4, v_6, v_8\}$ is a triangle.

Since $\{v_1, v_2, v_6, v_4\}$ forms a cycle of length four, without loss of generality, we may let

$$v_1 := (1, 1, 1, \dots, 1), \quad v_2 := (2, 1, 1, \dots, 1), \quad v_4 := (1, 2, 1, \dots, 1), \quad v_6 := (2, 2, 1, \dots, 1).$$

Since $\{v_1, v_2, v_3\}$ is a triangle, we have $v_3 = (3, 1, 1, \dots, 1)$. Since $\{v_1, v_4, v_5\}$ is a triangle, we have $v_5 = (1, 3, 1, \dots, 1)$. Since $\{v_2, v_6, v_7\}$ is a triangle, we have $v_7 = (2, 3, 1, \dots, 1)$. Since $\{v_4, v_6, v_8\}$ is a triangle, we have $v_8 = (3, 2, 1, \dots, 1)$. Since $\{v_3, v_8, v_9\}$ is a triangle, we have $v_9 = (3, 3, 1, \dots, 1)$. Then $\{v_5, v_7, v_9\}$ forms a triangle, and hence $G - v_{10}$ is isomorphic to H_1 in Figure 1. We complete the proof. \square

Theorem 3.4. For $n \geq 3$, we have $k(H(n, 3)) \geq (n - 3)3^{n-1} + 6$.

Proof. Suppose that $k(H(n, 3)) \leq (n - 3)3^{n-1} + 5$. Then, by Corollary 2.3, there exists an acyclic digraph D such that $C(D) = H(n, 3) \cup I_{(n-3)3^{n-1}+5}$, $N_D^-(v) \in \mathcal{F}(n, 3) \cup \{\emptyset\}$ for any $v \in V(D)$, and $|\{v \in V(D) \mid N_D^-(v) = \emptyset\}| = 5$. Let $v_1, v_2, \dots, v_{(n-3)3^{n-1}+5}$ be an acyclic ordering of D . We may assume that the vertices v_1, v_2, \dots, v_5 have no in-neighbors in D . For $6 \leq i \leq 12$, let $\mathcal{F}_i := \{N_D^-(x) \mid x \in \{v_1, v_2, \dots, v_i, v_{i+1}\}\}$ and let G_i be the subgraph of $H(n, 3)$ induced by $\{v_1, v_2, \dots, v_i\}$. Note that \mathcal{F}_i contains $i - 4$ triangles whose vertices are in $\{v_1, v_2, \dots, v_i\}$ and that G_i contains all the triangles in \mathcal{F}_i . Since G_{10} is a subgraph of $H(n, 3)$ with 10 vertices containing 6 triangles, by Lemma 3.3, G_{10} is isomorphic to $H_1 \cup I_1$ or H_2 , where H_1 and H_2 are the graphs drawn in Figure 1. Since G_{10} is a subgraph of G_{11} and G_{11} has 7 triangles, G_{10} must be isomorphic to H_2 and G_{11} must be isomorphic to H_3 in Figure 1. Then we can observe that any subgraph of $H(n, 3)$ with 12 vertices containing H_3 cannot have 8 triangles. However, G_{12} is a subgraph of $H(n, 3)$ with 12 vertices and 8 triangles, which is a contradiction. Hence $k(H(n, 3)) \geq (n - 3)3^{n-1} + 6$. \square

Proof of Theorem 1.3. Theorem 1.3 follows from Corollary 3.2 and Theorem 3.4. \square

4. Concluding Remarks

In this paper, we gave the exact values of the competition numbers of ternary Hamming graphs. Note that the bound given in Theorem 3.1 is tight when $q = 2$ or 3 , that is, $k(H(n, q)) = (n - q)q^{n-1} + k(H(q, q))$ holds for $n \geq q$ and $q \in \{2, 3\}$. We left a question for a further research whether or not the bound in Theorem 3.1 is tight for any $2 \leq q \leq n$.

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